Lecture 35

M diffeo $\mathbb{R}^n$ CAT(0)

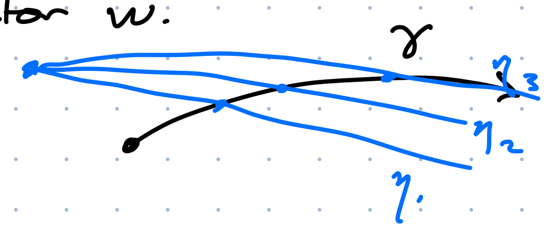
M Hadamard manifold. Geod ray $\gamma: \mathbb{R}^{\geq 0} \to M$ $\begin{cases} \|\gamma'\|=1 \\ \gamma' \text{ parallel} \\ \gamma(0) \text{ base pt} \end{cases}$

$\partial_{vis}(M) := \{\text{geod rays}\}/\sim$ $\quad \gamma_1 \sim \gamma_2$ if $\sup_t d(\gamma_1(t), \gamma_2(t)) < \infty$

Last time: $S(T_pM) \to \partial_{vis}(M)$ is injective.
$v \longmapsto [\exp(t\cdot v)]$

In fact it's also surjective. If γ ray at basept p, then let η_k be the ray from q that hits $\gamma(k)$.

Sketch. $\eta_k'(0)$ converges in T_qM, to vector w.
$[\exp(tw)] = [\gamma]$
Key: $d(\gamma_1(t), \gamma_2(t))$ convex. $\square$



∂_{vis} def by Eberlein, O'Neill.

There's a topology on $M \cup \partial_{vis}(M)$ that makes it a compact space.
Bridson-Haefliger II.8

We'll treat $\partial_{vis}(M)$ as a top space w/ $S(T_pM)$ topology.

Note. Homeo $S(T_pM) \to S(T_qM)$ is usually not smooth.
But it is smooth for symm spaces. (e.g. $\mathbb{H}^n$)

G acts on $\underbrace{\partial_{vis}(G/K)}_{\text{homeo to } S^{N-1}}$, but the action is somewhat subtle as $G \curvearrowright G/K$ changes base pts.

This action is by homeomorphisms.

Borel-J: Compactifications of symmetric & locally symmetric spaces

Recall $\mathfrak{a} \subset \mathfrak{p}$ max abelian subspace. Let $\lambda \in \mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$

$\mathfrak{g}_\lambda = \{v \in \mathfrak{g} \mid [h,v] = \lambda(h)v \ \forall h \in \mathfrak{a}\}$ root space.

$\rightsquigarrow$ root system $\Phi(\mathfrak{g}, \mathfrak{a})$, Weyl group $\cong N_K(\mathfrak{a})/Z_K(\mathfrak{a})$

The elts of $\mathfrak{g}$ that commute with $\mathfrak{a}$ have form

$$\underbrace{Z_{\mathfrak{g}}(\mathfrak{a})}_{\text{Levi}\ \mathfrak{h}} = \mathfrak{m} \oplus \mathfrak{a} \quad \text{for } \mathfrak{m} \subset \mathfrak{k}$$

Now introduce a notion of positivity, Φ^+ and Δ obtained.

$S \subset \Delta \rightsquigarrow \mathfrak{a}_S \subset \mathfrak{a} \qquad \mathfrak{a}_S = \bigcap_{\alpha \in S} \ker \alpha$

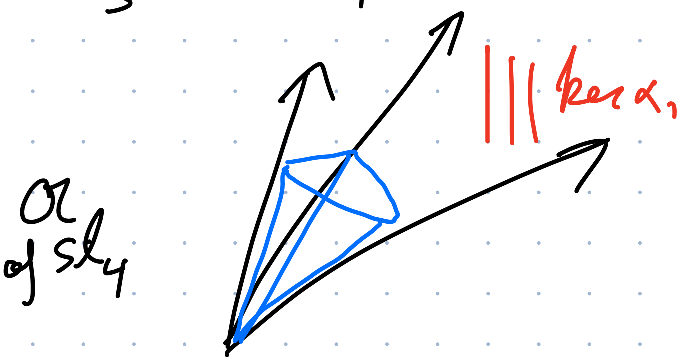
$\mathfrak{q}_S$ = subalg gen by $\underbrace{\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha}_{\mathfrak{n}}$ and $\overbrace{Z_{\mathfrak{g}}(\mathfrak{a})}^{\mathfrak{h}}$. and $\bigoplus_{\alpha \in S} \mathfrak{g}_{-\alpha}$

e.g. $\mathfrak{g} = \mathfrak{sl}_n \mathbb{R}$ then $\Phi = \Phi(\mathfrak{sl}_n \mathbb{C}, \mathfrak{h})$ and $\mathfrak{q}_S$ = block upper.

<u>Borel-Ji</u> Prop I.2.6 Let $\mu \in \partial_{vis}(G/K)$. Then $\mathrm{Stab}_G(\mu)$ is parabolic. Every parabolic is obtained this way.

$$\mathfrak{a}^+ = \{ t \in \mathfrak{a} \mid \lambda(t) > 0 \ \forall \lambda \in \Phi^+ \} = \{ t \in \mathfrak{a} \mid \lambda(t) > 0 \ \forall \lambda \in \Delta \}$$

$$\mathfrak{a}_S^+ = \{ t \in \mathfrak{a} \mid \lambda(t) \geq 0 \ \forall \lambda \in \Delta, \ = 0 \text{ iff } \lambda \in S \}$$



<u>Thm</u> Let $v \in \mathfrak{a}_S^+$. Then $P_S = \mathrm{Stab}(\exp(tv))$

In particular, the min parab is the Stab of $\exp(tv)$ for $v \in \mathfrak{a}^+$.

Let's examine this for $\mathfrak{sl}_n \mathbb{R}$.

$\lambda_1 > \dots > \lambda_n$.

$\mathfrak{a}$ = real diag sum zero. $\mathfrak{a}^+$ = decreasing $\quad v = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$